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ON THE BASIC LAWS OF FLUID-THERMODYNAMICS

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Consider a fluid in the $x = (x_1, x_2, x_3)$ - space and its motion in the time t. The velocity may be denoted by

$$u_k = D_t x_k, \quad u = (u_1, u_2, u_3) \quad q = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

The mass per unit volume, or density ρ may be considered a function of x and t, also considered unknown just as u.

Consider a region V with the boundary S. Then the continuity of mass requires that just the increase of mass per unit time equals the mass flowing in.

Let $n = (n_1, n_2, n_3)$ be the unit vector of the exterior normal, let $q_n = (u_1 n_1 + u_2 n_2 + u_3 n_3)$ be the normal component of velocity. Then the import of mass per unit time is

$$-\int_S \rho q_n \, dS,$$

hence

I $$D_t \int_V \rho \, dV + \int_S \rho q_n \, dS = 0$$

the differential form of which is

(1) $$\nabla_t \rho + \nabla_k (\rho u_k) = 0$$

with $$\nabla_t = \frac{\partial}{\partial t}, \quad \nabla_k = \partial/\partial x_k$$

or, with $D_t = \nabla_t + u_k \nabla_k$,

(1)' $$D_t \rho = -\rho \nabla_k u_k$$

Let p = p(x, t) be the pressure; let further the viscous stresses be given by the tensor

$$\{\sigma_{km}\} = \mu\left\{\nabla_k u_m + \nabla_m u_k - \tfrac{2}{3}\delta_{km}\nabla_\ell u_\ell\right\}$$

such that

$$\sigma_{\ell\ell} = 0,$$

μ being the viscosity coefficient. Then the force exerted against the surface S from the outside has the components

$$-\int_S p n_k \, dS + \int_S \sigma_{km} n_m \, dS$$

This force equals the increase of momentum per unit time of the mass that just happens to be in V at the time considered. This increase equals the increase of momentum per unit time contained in V diminished by the momentum imported per unit time into V. The latter is

$$-\int_S \rho u_k q_n \, dS$$

Hence the balance of force as momentum increase is

II
$$D_t \int_V \rho u_k \, dV + \int_S \rho u_k q_n \, dS = -\int_S p n_k \, dS + \int_S \sigma_{km} n_m \, dS$$

The differential form of this relation is

(2)
$$\nabla_t \rho u_k + \nabla_m \rho u_k u_m = -\nabla_k p + \nabla_m \sigma_{km}$$

or, by virtue of (1),

(2)'
$$\rho D_t u_k = -\nabla_k p + \nabla_m \sigma_{km}$$

To obtain an energy balance we start with the differential relation (2)' and multiply by u_k we obtain

$$\rho D_t \tfrac{1}{2} q^2 = -u_k \nabla_k p + u_k \nabla_m \sigma_{km} -$$

The left member, in view of (1), equals

$$\nabla_t \tfrac{1}{2}\rho q^2 + \nabla_m \tfrac{1}{2}\rho q^2 u_m$$

We apply integration by parts on the right hand member and make use of the relation

$$\nabla_m u_k \cdot \sigma_{km} = \tfrac{1}{2}(\nabla_m u_k + \nabla_k u_m)\,\sigma_{km} = \mu\varphi\,, \text{ where}$$

$$\varphi = \tfrac{1}{2}\left(\nabla_m u_k + \nabla_k u_m - \tfrac{2}{3}\delta_{mk}\nabla_\ell u_\ell\right)^2 > 0. \quad \times$$

$\mu\varphi$ is called the dissipation. Thus we obtain

(3) $$\rho D_t \tfrac{1}{2} q^2 = \nabla_t \tfrac{1}{2}\rho q^2 + \nabla_m \tfrac{1}{2}\rho q^2 u_m$$

$$= \nabla_k u_k p + p\nabla_\ell u_\ell + \nabla_m u_k \sigma_{km} - \mu\varphi. \quad \times$$

After integration we find the energy balance relation

$$D_t \int_V \tfrac{1}{2}\rho q^2\, dV + \int_S \tfrac{1}{2}\rho q^2 q_n\, dS$$

III $$= -\int_S p q_n\, dS + \int_V p\nabla_\ell u_\ell\, dV$$

$$+\int_S u_k \sigma_{km} n_m\, dS - \int_V \mu\varphi\, dV$$

The left member is the increase per unit time of the kinetic energy contained in V diminished by the import of kinetic energy per unit time or, what is the same thing, the increase per unit time of the kinetic energy of the mass that happens to be in V at the time considered. The right hand side consists of the power import (i.e. energy import per unit time) of pressure and viscous stresses augmented by the expansion work per unit time $\int_V p\nabla_\ell u_\ell\, dV$ done by the pressure and the total dissipation work per unit time $-\int_V \mu\varphi\, dV$ done by the viscous stresses inside the volume. It is to be noted that the dissipation work done by

the interior viscous stresses is negative. A slightly different interpretation of relation III is perhaps more suggestive. We write relation III in the form

III'
$$-\int_S pq + \int_S u_k \tau_{km} n_m \, dS$$
$$= D_t \int_V \tfrac{1}{2} \rho q^2 dV + \int_S \tfrac{1}{2} \rho q^2 q_n \, dS$$
$$- \int_V p \nabla_\ell u_\ell \, dV + \int_V \mu \varphi \, dV$$

and interpreted it as follows: The work import of the forces at the surface is partly used to increase the kinetic energy of the mass in V, partly lost by compression and by dissipation.

It is, of course, desirable to have the lost energy re-emerge as increase of an internal potential energy. This is possible on the basis of the principles of thermodynamics.

Let ϵ, s be the internal energy and entropy per unit mass and let t be the temperature. The energy is to be considered a function

$$\epsilon = \epsilon(\rho, s)$$

of density andentropy with the differential

$$d\epsilon = p \, d\rho^{-1} + \tau \, ds$$

Let further λ be the coefficient of heat conduction such that the heat energy transported per unit time across the surface S' in direction of n equals $-\lambda \nabla_n T = -\lambda \nabla_k T \, n_k$. Then the energy balance is that the lost energy equals the increase per unit time

of internal energy diminished by the heat conducted to the outside per unit time.

IV' $$-\int_V [illegible]\, dV = \int_V [illegible]$$

or $$[illegible]\, dV + \int_S \rho e\, \chi_n [illegible] = -\int_S \lambda \nabla_n T\, dS$$

IV $$[illegible] \int_V [illegible]\, dV + \int_S [illegible]\, \chi_n\, dS$$
$$= -\int_V p \nabla_k u_k\, dV + \int_V \mu q\, dV + \int_S \lambda \nabla_n T\, dS$$

i.e. the increase of internal energy per unit time equals the sum of the compression work done per unit time, of the dissipation, of the heat imported per unit time through the boundary. The differential form of IV is

(4) $$[illegible]$$

Making use of $\rho\, de = [illegible] + \rho T\, ds$

and using relation (1)' we obtain the relation

(5) $$\rho T D_t s = \mu q + \nabla_k \lambda \nabla_k T$$

which expresses that the increase of heat per unit volume per unit time is the sum of dissipation and the heat import per unit volume. It is essential that [illegible] differential relation [illegible] counterpart integral relation. It is to be noted that relation (5) implies the [illegible] law of thermodynamics, for if there is no heat conduction into the element of volume considered we have [illegible] since [illegible], that is, its entropy can never decrease.

When relations III and IV are added we obtain the relation

VI $$[illegible] \int_V [illegible] + \tfrac{1}{2} \chi^2 [illegible]\, dV + \int_S [illegible] + \tfrac{1}{2}\chi^2 [illegible]\, dS$$

$$= -\int_S p \chi_n\, dS + \int_S u_k \tau_{km} n_m\, dS + \int_S \lambda \nabla_n T\, dS$$

which states that the increase per unit time of the total energy

$$\int_V \rho\{\epsilon + \tfrac{1}{2}u^2\}\,dV$$

is the sum of the total work done per unit time against the boundary from the outside and the heat imported per unit time. Relation VI expresses that no energy is lost. Its differential form is

$$(6) \qquad \nabla_t\, \rho\{\epsilon + \tfrac{1}{2}u^2\} + \nabla_k\, \rho\{\epsilon + \tfrac{1}{2}u^2\} u_k$$

$$= \rho D_t\{\epsilon + \tfrac{1}{2}u^2\} = -\nabla_k\, p\, u_k + \nabla_i\, u_k \tau_{ki} + \nabla_k \lambda \nabla_k T.$$

It is natural to investigate the energy balance along a stream line. To this end it is convenient to introduce the enthalpy

$$i = \epsilon + \rho^{-1} p$$

which should be considered a function of p and s. Its differential is

$$di = \rho^{-1} dp + T\,ds.$$

From (1)' we have the relation

$$\rho D_t\, \rho^{-1} p = D_t\, p - \rho^{-1} p D_t\, \rho = D_t\, p + p \nabla_k u_k$$

$$= \nabla_t\, p + \nabla_k\, p u_k.$$

Hence we obtain from (6) the relation

$$(7)' \qquad D_t\{i + \tfrac{1}{2}u^2\} = \rho^{-1}\nabla_t\, p + \rho^{-1}\nabla_i\, u_k \tau_{ki} + \rho^{-1}\nabla_k \lambda \nabla_k T$$

which does not seem particularly useful.

A different aspect is obtained when a steady state is considered in which all quantities depend on x only. Then we consider as an volume V a stream tube with the entrance S_1, exit S_2 and the

lateral surface S_3 on which $\chi_n = 0$. We change the notation by letting n be the inner normal on the entrance S_1.

Relation (6) can then be written

VIII
$$\left[\int_S \left\{ i + \tfrac{1}{2}\chi^2 \right\} \rho \chi_n \, dS \right]_{(1)}^{(2)}$$

$$= \left[\int_S \left\{ u_k \sigma_{km} n_m + \lambda \nabla_n T \right\} dS \right]_{(1)}^{(2)}$$

$$+ \int_{S_3} \left\{ u_k \sigma_{km} n_m + \lambda \nabla_n T \right\} dS$$

To interpret this relation it is useful to observe that by relation I

$$\left[\int_S \rho \chi_n \, dS \right]_{(1)}^{(2)} = 0$$

Accordingly we introduce the mass flux per unit time

$$M = \int_{S_1} \rho \chi_n \, dS = \int_{S_2} \rho \chi_n \, dS$$

Further we introduce the averages

$$\widetilde{i + \tfrac{1}{2}\chi^2} = \int_S \left\{ i + \tfrac{1}{2}\chi^2 \right\} \rho \chi_n \, dS \Big/ \int_S \rho \chi_n \, dS$$

for $S = S_1$ and $S = S_2$

Then we have

$$\left[\widetilde{i + \tfrac{1}{2}\chi^2}\right]_{(1)}^{(2)} = \left[\frac{1}{M}\int_S \left\{ u_k \sigma_{km} n_m + \lambda \nabla_n T \right\} dS \right]_{(1)}^{(2)}$$

$$+ \frac{1}{M}\int_{S_3} \left\{ u_k \sigma_{km} n_m + \lambda \nabla_n T \right\} dS$$

That is: the increase of the average value of $i + \frac{1}{2}\chi^2$

equals the sum of the total work put in by the viscous forces per unit time divided by the mass influx per unit time and the total heat imported per unit time divided by the mass influx per unit time.